THE ANALYST.

VOL. II. MARCH, 1875. NO. 2.

THE USE OF IMAGINARY QUANTITIES IN INTEGRAL CALCULUS.

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We have seen in the ANALYST of Nov. and Dec. 1874, that the central equation of the ellipse may be written in this form

(1) $$z = \sin(\zeta + ia).$$

The major axis of this ellipse is $\cos ia$, or $\frac{1}{2}(e^a + e^{-a})$ and the minor axis is (numerically, i. e., regardless of direction) $\sin ia \div i$, or $\frac{1}{2}(e^a - e^{-a})$. The abscissa of any point of the ellipse is $\cos ia.\sin\zeta$ and the ordinate of the same point is (expressing its direction) $\sin ia.\cos\zeta$. The element of the area of the ellipse is therefore given by the expression

$$dA = \sin ai.\cos\zeta.d(\cos ai.\sin\zeta) = \cos ai.\sin ai.\cos^2\zeta\, d\zeta.$$

Hence, integrating

(2) $$A = \tfrac{1}{2}.\cos ai.\sin ai(\sin\zeta.\cos\zeta + \zeta)$$

which is the area of the ellipse from $\zeta = 0$ to $\zeta = \zeta$. This value of course appears here in an imaginary form, since according to our notation every ordinate is an imaginary quantity. *Numerically* we have

(2_a) $$A = [\cos ai.\sin ai\,(\sin\zeta.\cos\zeta + \zeta)] \div 2i$$

which is a "real" magnitude.

I will now show that the value of A in equation (2) is the same as that obtained by the ordinary method, namely, the central equation of the ellipse being $a^2y^2 + b^2x^2 = a^2b^2$, we find

$$A = \frac{b}{2a}\left[x\sqrt{(a^2 - x^2)} + a^2.\text{arc}\sin\frac{x}{a}\right].$$

Introducing the eccentric angle ϕ, so that $x = a.\sin\phi$; $y = a.\cos\phi$; the last value of A becomes

$$A = \frac{b}{2a}\left[a.\sin\phi\,.\,a.\cos\phi + a^2.\phi\right] = \frac{1}{2}ab\left[\sin\phi.\cos\phi + \phi\right]$$

which is identical with (2) for $a = \cos ai$; $b = \sin ai$ (numerically). We now see that ζ is the same as the "eccentric" angle ϕ. Let it be remembered that $\sin ai$ and $\sin ai \div i$ are identical so far as magnitude is concerned. The ordinary method does not express direction; but only magnitude.

In the same number of the ANALYST I have shown that the central equation of the hyperbola may be written thus

(3) $$z = \sin(a + i\zeta).$$

The axes are here respectively $\sin a$ and $\cos a$; the abscissa of any point is $\sin a.\cos i\zeta$ and the ordinate of the same point is, *numerically*, $\cos a \times \sin i\zeta \div i$. The element of the area here becomes

$$dA = \cos a.\sin i\zeta.d(\sin a.\cos i\zeta) = -\cos a.\sin a.\sin^2 i\zeta.id\zeta$$

which gives

(4) $$A = \cos a.\sin a.\tfrac{1}{2}(\sin i\zeta.\cos i\zeta - i\zeta)$$

as the area of the hyperbola. This equation is precisely the counterpart of eq. (2). By the ordinary method we find from the central equation $b^2x^2 - y^2a^2 = a^2b^2$, for A the expression

(4_a) $$A = \tfrac{1}{2}\left[xy - ab.\log\left(\frac{x}{a} + \frac{y}{b}\right)\right].$$

To compare this result with our own, I will again introduce the "eccentric angle" ψ, so that $x = a.\sec\psi$; $y = b.\tan\psi$, then the last value of A becomes

$$A = \tfrac{1}{2}[a.\sec\psi.b.\tan\psi - ab.\log(\sec\psi + \tan\psi)].$$

But since the abscissa x and the ordinate y according to our notation are $x = \sin a.\cos i\zeta$; $y = \cos a.\sin i\zeta \div i$ (absolutely) we find

$$a = \sin a;\ b = \cos a$$

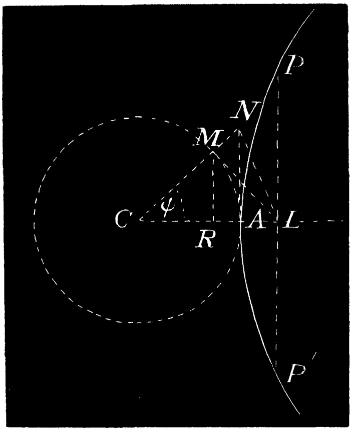
(5) $$\cos i\zeta = \sec\psi$$

(6) $$\sin i\zeta \div i = \tan\psi$$

as necessary conditions that the two values of A become identical. For, now the last value of A becomes

$$A = \tfrac{1}{2}ab[\cos i\zeta.\sin i\zeta \div i - \log(\cos i\zeta + \sin i\zeta \div i)].$$

But $\cos i\zeta + \sin i\zeta \div i = e^{\zeta}$; $\log(\cos i\zeta + \sin i\zeta \div i) = \zeta$; which shows that *numerically* (4) and (4_a) are identical. For, the numerical value of A in (4) is $A = \cos a.\sin a.\frac{1}{2}(\sin i\zeta.\cos i\zeta \div i - \zeta)$.



Equations (5) and (6) establish a simple relation between the circular functions of an imaginary arc $i\eta$ and those of a corresponding real arc ψ; and they give rise to a handsome construction, which leads us to understand the nature of the imaginary circular functions.

Let PAP' be an hyperbola, whose central equation is $x^2b^2 - y^2a^2 = a^2b^2$; then, introducing the eccentric angle $ACM = \psi$ and drawing the tangent LM, we have

$$x = a.\sec\psi; \qquad y = b.\tan\psi.$$

But from our equation of the hyperbola in connection with these last two equations we found the equations (5) and (6), which shows that, putting

$a = 1, \qquad \cos i\eta = CL = CN; \qquad \sin i\eta \div i = AN;$

which last expression is of course again taken in the absolute sense, i. e., regardless of direction. And since

$\tan i\eta = \dfrac{\sin i\eta}{\cos i\eta}$ we also find $\tan i\eta = \dfrac{i.\tan\psi}{\sec\psi} = i.\sin\psi$; $\dfrac{\tan i\eta}{i} = MR.$

Thus we have established the following relations:

(5) $$\cos i\eta = \sec\psi = \frac{1}{\cos\psi}$$

(6) $$\frac{\sin i\eta}{i} = \tan\psi$$

(7) $$\frac{\tan i\eta}{i} = \sin\psi$$

(8) $$dx = \frac{d\psi}{\cos\psi}; \quad d\psi = \frac{dx}{\cos ix}.$$

These four equations are sufficient for a complete discussion of all the circular functions of an imaginary argument. Let it be observed that the argument $i\eta$ does not appear in the figure; but only its functions $\cos i\eta$, $\sin i\eta$ and $\tan i\eta$ which depend upon the angle ψ. These four equations evidently hold for all values of ψ and of x. Hence we have the following

Theorem:—To every imaginary angle $i\eta$, there corresponds a cognate real angle ψ; and all circular functions of $i\eta$ can be expressed by means of circular functions of ψ and vice versa.

It is well known from the theory of complex functions that all relations obtaining for the circular functions of a real argument hold equally for the like functions of an imaginary and of a complex argument. Also that all the rules of differentiation and integration hold for imaginary functions. Hence we are warranted in deducing Nos. (7) and (8).

If in the equation of the parabola we put $a = b = 1$, then $x^2 - y^2 = 1$, and this is the central equation of the equilateral hyperbola. Again employing our imaginary function we have

$$z = \cos i\eta + \sin i\eta$$

as the equation of the same curve. Now $\cos i\eta$ being the abscissa and $\sin i\eta$ the ordinate of this curve, we see that this curve gives immediately the values of $\cos i\eta$ and $\sin i\eta$. A slight modification changes these two functions into what is well known as the *hyperbolic* cosine and sine of η; namely

$$\cos i\eta = \cos \text{hyp.}\eta; \qquad \sin i\eta \div i = \sin \text{hyp.}\eta.$$

But *Gudermann*, *Lambert* and their followers did not express the condition that cos.hyp.x and sin.hyp.x are perpendicular to each other (in our figure CN and AN) and thus they arrived at the fundamental equation

$$(\cos.\text{hyp}.x)^2 - (\sin.\text{hyp}.x)^2 = 1$$

whilst our own notation gives us

$$(\cos i\xi)^2 + (\sin i\xi)^2 = 1$$

which at once reduces the so-called hyperbolic functions to circular functions. In fact this last formula together with the well known equation

$$\sin(ix \pm iy) = \sin ix.\cos iy \pm \cos ix.\sin iy$$

is sufficient to show that all the equations which hold for the circular functions of real arcs, are equally applicable to the circular functions of imaginary arcs. Now we have the following

Theorem:—The so-called hyperbolic functions of a variable ξ are numerically identical with the corresponding circular functions of $i\xi$; and all goniometric formulæ apply to the hyperbolic functions.

The reason why the hyperbolic functions have not come into general use is, undoubtedly, the fact that formulæ expressing their relations to each other did not agree with the circular formulæ.

I will now present two very convenient formulæ for the numerical computation of $\sin i\xi$ and $\cos i\xi$. Employing the letter x instead of ξ we have, by expanding the exponential equivalent of $\sin ix$

$$\sin ix = i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\right].$$

Comparing this with

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

we find

$$\frac{\sin ix}{i} = 2\left[x + \frac{x^5}{5!} + \frac{x^9}{9!} + \ldots\right] - \sin x \tag{9}$$

a series which converges very rapidly and of which two terms afford all desirable accuracy at least as far as $x = 1$. In a similar manner we find

$$\cos ix = 2\left[1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \ldots\right] - \cos x. \tag{10}$$

By employing these two formulæ I have calculated the following little table of imaginary circular functions, which will be acceptable, since the tables of hyperbolic functions are not generally accessible.

TABLE OF CIRCULAR FUNCTIONS OF AN IMAGINARY ARGUMENT.

Deg.	Arc x	$\cos ix$	$\frac{1}{i}\sin ix$	$\frac{1}{i}\tan ix$	Deg.	Arc x	$\cos ix$	$\frac{1}{i}\sin ix$	$\frac{1}{i}\tan ix$
1	0.0174	1.0001	0.0174	0.0174	50	0.8726	1.4051	0.9877	0.70292
5	0.0872	1.0038	0.0873	0.0870	55	0.9599	1.4972	1.1178	0.74661
10	0.1745	1.0152	0.1754	0.1727	60	1.0472	1.6002	1.2494	0.78074
15	0.2618	1.0344	0.2648	0.2558	65	1.1344	1.7155	1.3939	0.81255
20	0.3490	1.0615	0.3555	0.3349	70	1.2217	1.8472	1.5491	0.83865
25	0.4363	1.0967	0.4503	0.4106	75	1.3090	1.9896	1.7179	0.86344
30	0.5236	1.1402	0.5478	0.4805	80	1.3963	2.1524	1.9005	0.88298
35	0.6108	1.1924	0.6495	0.5447	85	1.4835	2.3212	2.0928	0.90162
40	0.6981	1.2566	0.7562	0.6016	90	1.5708	2.5085	2.3002	0.91740
45	0.7854	1.3245	0.8683	0.6555	95	1.6581	2.7189	2.5283	0.92987
					100	1.7453	2.9500	2.7757	0.94092

Before proceeding to the further application of our imaginary circular function, I will present two elementary integrals already known from the theory of hyperbolic functions. Putting arc sin $ix = iy$, we find

$$\frac{i.dx}{\sqrt{(1-i^2x^2)}} = i.dy, \text{ hence } \frac{dx}{\sqrt{(1+x^2)}} = dy, \text{ and}$$

(11) $$\int\frac{dx}{\sqrt{(1+x^2)}} = \frac{1}{i}\text{ arc sin } ix.$$

Again, putting arc tan $ix = iy$, we find

$$\frac{i.dx}{1+i^2x^2} = i.dy, \text{ hence}$$

(12) $$\int\frac{dx}{1-x^2} = \frac{1}{i}\text{ arc tan } ix.$$

These two formulæ will serve as elementary integrals precisely as their well known counterparts

$$\int\frac{dx}{\sqrt{(1-x^2)}} = \text{arc sin } x, \text{ and } \int\frac{dx}{1+x^2} = \text{arc tan } x.$$

For the integration of circular functions it is often quite advantageous to introduce the *cognate imaginary* angle. For example let the integral

$$\int\frac{dx}{\cos^4 x} \text{ be required.}$$

Introducing the *cognate* angle $i\phi$, we have from (8) and (5)

$$\int\frac{dx}{\cos^4 x} = \int\left(\frac{d\phi}{\cos i\phi} \div \frac{1}{\cos^4 i\phi}\right) = \int\cos^3 i\phi d\phi = \int(1-\sin^2 i\phi)\cos i\phi d\phi$$
$$= (1 \div i)(\sin i\phi - \tfrac{1}{3}\sin^3 i\phi).$$

Again, to integrate $\int\frac{dx}{\cos^6 x}$ we have $\int\left(\frac{d\psi}{\cos i\psi}\div\frac{1}{\cos^6 i\psi}\right)$

$$=\int\cos^5 i\psi d\psi=\int(1-2\sin^2 i\psi+\sin^4 i\psi)\cos i\psi d\psi$$

$$=\frac{1}{i}\left(\sin i\psi-\tfrac{2}{3}\sin^3 i\psi+\tfrac{1}{5}\sin^5 i\psi\right).$$

For the integration of algebraic irrational functions formula (11) ts often quite useful, and (12) for rational algebraic expressions, as is known from the theory of hyperbolic functions.

All rational functions of $\sin x$ and $\cos x$, by the introduction of their exponential equivalents, can be reduced to rational functions of exponentials. For, let it not be supposed that $\frac{1}{2}(e^{ix}+e^{-ix})$ and $\frac{1}{2}(e^{ix}-e^{-ix})\div i$ are imaginary. Both these expressions are "real", as is readily seen when expanding them; and they will always lead to "real" results when their goniometric equivalents will do so.

I will now give an example of the application of imaginary circular functions to the rectification of curves. The equation

$$z=\cos ix+\sin ix$$

represents an equilateral hyperbola. It is the central equation of that curve, usually represented by $x^2-y^2=1$. According to our notation the abscissa of any point of the curve is $\cos ix$ and the ordinate $\sin ix$. Hence the element of the length of the curve is expressed by

$$ds=\sqrt{[(d.\cos ix)^2+(d.\sin ix)^2]}.$$

But what we want is the *numerical* value of s, so that the squares of both differentials must be positive, hence

$$(13)\qquad ds=\sqrt{(-\sin^2 ix+\cos^2 ix)}dx,$$

for, $\sin ix$ is imaginary, hence $\sin^2 ix$ negative, and $-\sin^2 ix$ positive; also $\cos ix$ is real, hence $\cos^2 ix$ positive. This last equation may also be written thus $\quad ds=\sqrt{(\cos 2ix)}dx=\sqrt{[\frac{1}{2}(e^{2x}+e^{-2x})]}dx=e^x.\sqrt{\tfrac{1}{2}}$

$$\times\sqrt{\left[1+\frac{1}{e^{4x}}\right]}dx=e^x.\sqrt{\tfrac{1}{2}}\left[1+\frac{1}{2}.\frac{1}{e^{4x}}-\frac{1}{2}\frac{1}{4}.\frac{1}{e^{8x}}+\frac{1}{2}\frac{1}{4}\frac{3}{6}.\frac{1}{e^{12x}}-.\right]dx$$

Hence, integrating

$$(14)\qquad s=\frac{1}{\sqrt{2}}\left[e^x-\frac{1}{2}\frac{1}{3}.\frac{1}{e^{3x}}+\frac{1}{2}\frac{1}{4}.\frac{1}{7e^{7x}}-\frac{1}{2}\frac{1}{4}\frac{3}{6}.\frac{1}{11e^{11x}}+.\right]-C$$

which series converges very rapidly, so that a few terms of it will almost always afford all desired accuracy. To determine the constant C, we will put $s=0$, so that $x=0$, hence

$$C=\sqrt{\frac{1}{2}}\cdot\left[1-\frac{1}{2}\frac{1}{3}+\frac{1}{2}\frac{1}{4}\frac{1}{7}-\frac{1}{2}\frac{1}{4}\frac{3}{6}\frac{1}{11}+\ldots\right]$$

which gives $C=0.59922$.

Now if x be taken equal to 2, yet for this small number the first two terms of the series will afford tolerable accuracy. Let $x=2$, which corresponds to the abscissa $\cos ix=3.7711$, or to the abscissa counted from the vertex $=2.7711$; then $s=\sqrt{\tfrac{1}{2}}.(7.38905-\tfrac{1}{6}.\tfrac{1}{403.42})-0.59922=4.6175$.

As x increases the value of s approaches more and more to

$$s=e^x.\sqrt{\tfrac{1}{2}}-0.59922.$$

In a similar manner the rectification of the hyperbola whose equation is $a^2x^2-b^2y^2=a^2b^2$ may be performed. Also the rectification of the ellipse. Putting $z=\sin(a+ix)$ we have the equation of the hyperbola, as before. The element of the curve here becomes

$$ds=\sqrt{(\cos^2 ix-\tan^2 a\sin^2 ix)}dx$$

$$=\sqrt{[\tfrac{1}{4}(e^{2x}+2+e^{-2x})+\tfrac{1}{4}\tan^2 a(e^{2x}-2+e^{-2x})}dx$$

$$=\tfrac{1}{4}e^x\sqrt{\left[1+\frac{c}{e^{2x}}+\frac{1}{e^{4x}}\right]}dx,$$ denoting by c the quantity $2\left[\frac{1-\tan^2 a}{1+\tan^2 a}\right]$

Hence $ds=\tfrac{1}{4}e^x\left[1+\frac{1}{2}\left(\frac{c}{e^{2x}}+\frac{1}{e^{4x}}\right)-\frac{1}{2}\frac{1}{4}\left(\frac{c}{e^{2x}}+\frac{1}{e^{4x}}\right)^2+\ .\ .\ \right]dx$

and integrating

$$(15)\quad s=\frac{1}{4}\left[e^x-\frac{c}{2}\cdot\frac{1}{e^x}-\frac{1}{2}\frac{1}{3}\cdot\frac{1}{e^{3x}}+\frac{1}{2}\frac{1}{4}\left(\frac{c^2}{3e^{3x}}+\frac{2c}{5e^{5x}}+\frac{1}{7e^{7x}}\right)+\ .\ .\ \right]$$

which series also converges very rapidly.

In conclusion I will state that all integrals of the form $\int\cos^m x\sin^n x\,dx$ and most of the binomial integrals $\int x^m\left(a+bx^n\right)^{\frac{p}{q}}dx$ can be found in a more simple and elegant way than taught in the text-books, by employing imaginary circular functions.

Paradox for Students in Analytical Geometry. — The equation of a given curve of the third degree being

$$Ax^3+Bx^2y\ldots+Ex^2+Fxy\ldots+Hx\ldots+K=0,$$

if $P_1,(x_{\prime}y_{\prime})$, is a point on the curve, the equation

$$Ax_{\prime}^2x+\tfrac{1}{3}B(2x_{\prime}y_{\prime}x+x_{\prime}^2y)\ldots+\tfrac{1}{3}E(x_{\prime}^2+2x_{\prime}x)+\tfrac{1}{3}F(x_{\prime}y_{\prime}+x_{\prime}y+xy_{\prime})$$
$$\ldots+\tfrac{1}{3}H(x_{\prime}+2x)\ldots+K=0\ldots(1)$$

represents the tangent to the curve at the point P_1.